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A Note on the Integral Equation of the First Kind with a Cauchy Kernel

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PREPARED UNDER
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WITH THE
OFFICE OF NAVAL RESEARCH

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IMM-NYU 301
July 1962

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A NOTE ON THE INTEGRAL EQUATION OF THE FIRST KIND WITH A CAUCHY KERNEL

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There are many physical problems which can be reduced to the solution of the equation

$$(1) \qquad \int_a^b \frac{\Phi(t)dt}{t-\tau} = F(\tau) \ , \qquad a < \tau < b$$

where $F(\tau)$ is a prescribed function; and it has been recognized for well over thirty years that this equation is of central importance for the theory of singular integral equations. Several methods have been devised to find the solution of (1) in closed form. These methods can be classified according to the fundamental mathematical ideas which are used to obtain the solution:

1. Methods which involve the use of techniques which have their roots in the theory of analytic functions of a complex variable. [1], [2], [3].

2. Methods which resort to the Fourier theory of expansions in terms of orthonormal functions. [4], [5].

3. Methods which employ the Hardy-Poincaré-Bertrand formula. [1], [3].

4. Methods which depend on transform theory without the use of function theory techniques.

The author has not seen the fourth method in the literature but many readers will be able to reproduce this method with the clue that after the transformation $t = (b+a\xi)/(\xi+1)$, $\tau = (b+ax)/(x+1)$ which transforms the interval (a,b) to $(\infty,0)$; then (without the use of the Mellin inversion formula, and without the use of function theory techniques) equation (1) can be reduced to the solution of

$$\int_0^\infty \xi^s \psi(\xi) d\xi = 0$$

which under appropriate assumptions possesses only the trivial solution $\psi(\xi) \equiv 0$.

We are concerned here with the presentation of another method which leads to a new formula for the solution of (1). The method is in some sense more elementary than any of the methods mentioned above, and it consequently may have some pedagogical value. We proceed to show that (1) can be reduced to the solution of Abel's integral equation. The reduction is an example of the application of a known idea (probably under exploited) which is described below.

A simple translation shows that (1) can be replaced by

$$\int_0^1 \frac{\phi(\xi) d\xi}{\xi - x} = f(x) , \qquad 0 < x < 1 \tag{2}$$

without loss of generality. We assume for simplicity that $\phi(\xi)$ satisfies a uniform Hölder condition for any pair of values in the interior of the closed interval [0,1] and thus guarantee the existence of the Cauchy principal value of the integral in (2). Also, at an end point α we admit that $\phi(\xi)$ may have a singularity like $\ell n |\alpha - \xi|$, or $1/|\alpha - \xi|^\gamma$ where $\gamma < 1$. Furthermore, we suppose that the prescribed function f(x) is a member of the class of functions to which $\phi(\xi)$ belongs. Let us prepare the equation (2) by writing it in the form

$$(3) \qquad \int_0^1 \frac{\xi\phi(\xi)d\xi}{\xi - x} = xf(x) + c$$

and then

$$(4) \qquad \int_0^1 \frac{\xi\phi(\xi)d\xi}{[(\sqrt{\xi})^2 - (\sqrt{x})^2]\sqrt{x}} = \sqrt{x}\ f(x) + \frac{c}{\sqrt{x}}$$

where

$$c = \int_0^1 \phi(\xi)d\xi \ .$$

Integration of (4) gives

$$(5) \qquad -\int_0^1 \ell n\left|\frac{\sqrt{\xi} - \sqrt{x}}{\sqrt{\xi} + \sqrt{x}}\right| \sqrt{\xi}\ \phi(\xi)d\xi = \int_0^x \sqrt{\lambda}\ f(\lambda)d\lambda + 2c\sqrt{x} \ .$$

The kernel of this equation can be expressed as

$$(6) \qquad -\ell n\left|\frac{\sqrt{\xi} - \sqrt{x}}{\sqrt{\xi} + \sqrt{x}}\right| = \begin{cases} \displaystyle\int_0^\xi \frac{d\sigma}{\sqrt{\xi - \sigma}\ \sqrt{x - \sigma}} & x > \xi \\ \displaystyle\int_0^x \frac{d\sigma}{\sqrt{\xi - \sigma}\ \sqrt{x - \sigma}} & \xi > x \ . \end{cases}$$

If this representation is substituted in (5) we find

$$(7) \qquad \int_0^x \sqrt{\xi}\ \phi(\xi) \int_0^\xi \frac{d\sigma}{\sqrt{\xi - \sigma}\ \sqrt{x - \sigma}}\, d\xi + \int_x^1 \sqrt{\xi}\ \phi(\xi) \int_0^x \frac{d\sigma}{\sqrt{\xi - \sigma}\ \sqrt{x - \sigma}}\, d\xi$$
$$= \int_0^x \sqrt{\lambda}\ f(\lambda)d\lambda + 2c\sqrt{x}$$

and after changing the order of integration in each of the integrals to the left of the equality sign in (7) we have

$$(8)\qquad \int_0^x \frac{1}{\sqrt{x-\sigma}} \int_\sigma^x \frac{\sqrt{\xi}\,\phi(\xi)d\xi}{\sqrt{\xi-\sigma}}\,d\sigma + \int_0^x \frac{1}{\sqrt{x-\sigma}} \int_x^1 \frac{\sqrt{\xi}\,\phi(\xi)d\xi}{\sqrt{\xi-\sigma}}\,d\sigma$$

$$= \int_0^x \sqrt{\lambda}\, f(\lambda)d\lambda + 2c\sqrt{x}$$

which yields

$$(9)\qquad \int_0^x \frac{1}{\sqrt{x-\sigma}} \int_\sigma^1 \frac{\sqrt{\xi}\,\phi(\xi)d\xi}{\sqrt{\xi-\sigma}}\,d\sigma = \int_0^x \sqrt{\lambda}\, f(\lambda)d\lambda + 2c\sqrt{x}\ .$$

It is well known that the solution of Abel's equation

$$(10)\qquad \int_0^x \frac{\psi(\sigma)}{\sqrt{\sigma - x}}\,d\sigma = g(x)$$

is

$$\psi(\sigma) = \frac{1}{\pi}\frac{d}{d\sigma}\int_0^\sigma \frac{g(x)dx}{\sqrt{\sigma - x}}$$

and if g(x) is differentiable

$$\psi(\sigma) = \frac{1}{\pi}\left[\frac{g(0)}{\sqrt{\sigma}} + \int_0^\sigma \frac{g'(x)dx}{\sqrt{\sigma - x}}\right]\ .$$

The application of this result to (9) leads to

$$(11)\qquad \int_\sigma^1 \frac{\sqrt{\xi}\,\phi(\xi)d\xi}{\sqrt{\xi-\sigma}} = \frac{1}{\pi}\left[\int_0^\sigma \frac{\sqrt{x}\,f(x)dx}{\sqrt{\sigma - x}} + c\int_0^\sigma \frac{dx}{\sqrt{x}\,\sqrt{\sigma - x}}\right]$$

$$= \frac{1}{\pi}\int_0^\sigma \frac{\sqrt{x}\,f(x)dx}{\sqrt{\sigma - x}} + c\ .$$

The solution of the integral equation

$$\int_{\sigma}^{1} \frac{\psi(\xi)d\xi}{\sqrt{\xi-\sigma}} = h(\sigma) \tag{12}$$

can easily be deduced from Abel's equation (10). It is

$$\psi(\xi) = -\frac{1}{\pi}\frac{d}{d\xi}\int_{\xi}^{1} \frac{h(\sigma)}{\sqrt{\sigma-\xi}}\,d\sigma \ .$$

From this we see that the solution of (11) is

$$\begin{aligned} \sqrt{\xi}\,\phi(\xi) &= -\frac{1}{\pi}\frac{d}{d\xi}\int_{\xi}^{1}\frac{cd\sigma}{\sqrt{\sigma-\xi}} - \frac{1}{\pi^2}\frac{d}{d\xi}\int_{\xi}^{1}\frac{1}{\sqrt{\sigma-\xi}}\int_{0}^{\sigma}\frac{\sqrt{x}\,f(x)dx}{\sqrt{\sigma-x}}\,d\sigma \\ &= \frac{c}{\pi\sqrt{1-\xi}} - \frac{1}{\pi^2}\frac{d}{d\xi}\int_{\xi}^{1}\frac{1}{\sqrt{\sigma-\xi}}\int_{0}^{\sigma}\frac{\sqrt{x}\,f(x)dx}{\sqrt{\sigma-x}}\,d\sigma \ . \end{aligned} \tag{13}$$

This expression may be regarded as a formula for the solution of (2) but it is not the standard one. The standard formula for the solution can be obtained from (13) if we change the order of integration to get

$$\sqrt{\xi}\,\phi(\xi) = \frac{c}{\pi\sqrt{1-\xi}} - \frac{1}{\pi^2}\frac{d}{d\xi}\left[\begin{aligned} &\int_{\xi}^{1}\sqrt{x}\,f(x)\int_{x}^{1}\frac{d\sigma}{\sqrt{\sigma-\xi}\,\sqrt{\sigma-x}}\,dx \\ +&\int_{0}^{\xi}\sqrt{x}\,f(x)\int_{\xi}^{1}\frac{d\sigma}{\sqrt{\sigma-\xi}\,\sqrt{\sigma-x}}\,dx \end{aligned}\right] \tag{14}$$

and then use

$$\ell n\left|\frac{\sqrt{1-\xi}+\sqrt{1-x}}{\sqrt{1-\xi}-\sqrt{1-x}}\right| = \begin{cases} \displaystyle\int_{0}^{1-x}\frac{du}{\sqrt{1-\xi-u}\,\sqrt{1-x-u}} = \int_{x}^{1}\frac{d\sigma}{\sqrt{\sigma-\xi}\,\sqrt{\sigma-x}} & x > \xi \\ \displaystyle\int_{0}^{1-\xi}\frac{du}{\sqrt{1-\xi-u}\,\sqrt{1-x-u}} = \int_{\xi}^{1}\frac{d\sigma}{\sqrt{\sigma-\xi}\,\sqrt{\sigma-x}} & \xi > x . \end{cases}$$

In this way we find that

$$\sqrt{\xi}\,\phi(\xi) = \frac{c}{\pi\sqrt{1-\xi}} - \frac{1}{\pi^2}\frac{d}{d\xi}\int_0^1 \ell n\left|\frac{\sqrt{1-\xi}+\sqrt{1-x}}{\sqrt{1-\xi}-\sqrt{1-x}}\right| \sqrt{x}\, f(x)dx$$

from which

$$\sqrt{\xi}\,\phi(\xi) = \frac{c}{\pi\sqrt{1-\xi}} - \frac{1}{\pi^2}\int_0^1 \frac{\sqrt{x}\,\sqrt{1-x}\; f(x)dx}{\sqrt{1-\xi}\,(x-\xi)}$$

or

$$(15) \qquad \phi(\xi) = -\frac{1}{\pi^2\sqrt{\xi(1-\xi)}}\left[\int_0^1 \frac{\sqrt{x(1-x)}f(x)dx}{x-\xi} - \pi\int_0^1 \phi(x)dx\right]$$

and this is the standard formula.

It should be observed that the method we have applied to find the solution of (2) is a particular case of a more general method which can be used to solve a Fredholm equation of the first kind

$$(16) \qquad \int_0^b K(x,\xi)\phi(\xi)d\xi = f(x)$$

when the kernel can be represented in the form

$$(17) \qquad K(x,\xi) = \begin{cases} \displaystyle\int_0^{\xi} K_1(x,\sigma)K_2(\xi,\sigma)d\sigma & \xi < x \\ \displaystyle\int_0^{x} K_1(x,\sigma)K_2(\xi,\sigma)d\sigma & x < \xi\,. \end{cases}$$

If this is possible, (16) can be reduced to

$$(18) \qquad \int_0^x K_1(x,\sigma) \int_\sigma^b K_2(\xi,\sigma)\phi(\xi)d\xi d\sigma = f(x)$$

and thus to a consideration of two Volterra equations of the first kind.

Our example shows that even though the prescribed kernel in (16) cannot be represented in the form (17) it may be that certain preliminary operations performed on (16) will transform (16) into an equation with a kernel which does have a representation (17).

We note in closing that there is at least one general type of integral equation for which a methodical technique for passing from (16) to (18) is known. If the difference kernel $K(x-\xi)$ satisfies certain conditions, then the important Wiener-Hopf equation

$$\int_0^\infty K(x-\xi)\phi(\xi)d\xi = f(x) \qquad 0 < x$$

can be converted into the form

$$\int_x^\infty K_1(\sigma-x) \int_0^\sigma K_2(\sigma-\xi)\phi(\xi)d\xi d\sigma = f(x) \qquad 0 < x$$

by using the method described in [6], p. 230. The discovery of methods for converting other general types of integral equations into the form (18) would be of considerable interest.

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